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PROBLEM OF AN EQUIVALENT ROD IN NONLINEAR THEORY OF SPRINGS*

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A spring with a large number of coils can approximately be considered as a one-dimensional continuum (an equivalent rod), whose particles are the coils of the spring. The problem of an equivalent rod is to construct the equations describing this continuum and to calculate the characteristics of the equivalent rod by means of the coil geometry and the elastic properties of the spring material. The problem of an equivalent rod has been investigated thoroughly in linear theory /1– 6/. In the geometrically nonlinear theory it remains substantially open (only certain exact solutions of problems on the tension, torsion, and bending of springs were known /7– 10/).

The problem of an equivalent rod is solved below by a variational-asymptotic method /11– 14/.

1. Theory of inextensible rods /7,14/. In the classical theory, a rod is modeled by a curve Γ, equipped with an ortho reference triad τ_a $(a, b, c = 1, 2, 3)$, whose vector τ_3 is tangent to Γ. The curve Γ can be considered as the middle line of a rod, while the plane swept by the vector τ_α can be considered as the plane of the transverse section of the rod. The rod state of strain is given by the components $r^i(\xi)$ of the radius-vector of points of the curve Γ and by the components $\tau_a^{\,i}(\xi)$ of the vectors τ_a ($\xi \in [0, l]$ is the arclength along Γ, the superscripts i, j, k correspond to projections on the axes of the Cartesian coordinate system of the observer x^i and run through the values 1,2,3; the quantities with super- and subscripts agree; the site of the index is selected in conformity with the rule of summation over repeated sub- and superscripts). It is assumed that the coordinate system of the observer and the ortho reference triad τ_a have identical orientation, and the determinant of the orthogonal matrix $\|\tau_a^{\,i}\| = +1$.

The quantities r^i and $\tau_a^{\,i}$ satisfy the constraints

$$dr^i/d\xi = \tau_3^{\,i}, \quad \tau_a^{\,i}\tau_{ib} = \delta_{ab} \tag{1.1}$$

where δ_{ab} is the Kronecker delta. The state of deformed of an inextensible rod has three functionally independent degrees of freedom.

The curvature and torsion of the curve Γ is characterized by the quantities $\omega_a = {}^1/_2 e_{abc}\tau_{,\xi}^{ib}\tau_i^{\,c}$, where e_{abc} is the Levi-Civita symbol, the comma before the ξ in the subscript denotes differentiation with respect to ξ.

The measures of the strain of an inextensible rod Ω_a are introduced by the equalities

$$\Omega_a = \omega_a - \omega_{(0)a} \tag{1.2}$$

The subscript (0) denotes the value of quantities in the unstrained state. Rod bending is characterized by the quantities $\Omega_\alpha (\alpha, \beta, \gamma, \delta = 1, 2)$, and the torsion by the quantity Ω_3.

When r^i and $\tau_a^{\,i}$ are given at the ends of the rod, the true position of the rod is the stationary point of the functional /14/

$$I = \int_0^l {}^1/_2 I^{ab}\Omega_a\Omega_b d\xi \tag{1.3}$$

where I^{ab} is the cross-section stiffness tensor.

2. Springs. A spring is an elastic rod with a special undeformed state. We will describe these states (in this section the consideration is for the undeformed state; to avoid awkwardness in all the formulas the subscript 0 referring to the undeformed state will be omitted).

Let there be a certain space curve $\bar\Gamma$ given by the equations $x^i = \bar r^i(\zeta)$, where ζ is a natural parameter along $\bar\Gamma$, and provided with an ortho reference triad $\bar\tau_a(\zeta)$ such that $\bar\tau_3^{\,i} = d\bar r^i/d\zeta$. Let us represent the radius-vector of the rod middle line Γ from which the spring is coiled in the form

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$$r^i(\xi) = \bar r^i(\zeta) + \rho^a(\zeta)\,\bar\tau_a{}^i(\zeta), \quad \zeta = \zeta(\xi) \tag{2.1}$$

Here $\rho^a(\xi)$ and $\zeta(\xi)$ are functions satisfying the condition $dr^i/d\xi\, dr_i/d\xi = 1$. The quantities $\rho^a(\xi)$ have the meaning of projections of the local radius vector $r^i - \bar r^i$ on the ortho reference triad vector $\bar\tau_a$.

Let us introduce the projections α_{ab} of the vectors τ_a with which the rod middle line is provided, by the vector $\bar\tau_b$

$$\tau_a{}^i = \alpha_a{}^b \bar\tau_b{}^i \tag{2.2}$$

Just as all the orthogonal matrices to be encountered later, the orthogonal matrix $\|\alpha_b{}^a\|$ has a determinant equal to 1 by definition.

Formulas (2.1) and (2.2) can be written for the undeformed state of an arbitrary rod.

We call an elastic rod a spring if functions $\bar r^i(\zeta)$ and $\bar\tau_a{}^i(\zeta)$ exist such that the functions $\rho^a(\xi), \alpha_{ab}(\xi)$ and the physical characteristics $I^{ab}(\xi)$ are represented in the form of a function $f(\eta, \xi)$ of a fast variable η and a slow variable ξ (where $\eta = \eta(\xi)$) with the following properties: 1) the function f is periodic in η with period one: 2) $d\eta/d\xi \equiv 1/\Delta(\xi) \geqslant \mathrm{const} > 0$, $\Delta \ll l$; 3) the characteristic scale L of variation of the function $f(\eta, \xi)|_{\eta=\mathrm{const}}$ and $\Delta(\xi)$ in ξ satisfy the condition $\Delta \ll L$. Moreover, for springs the functions $\varkappa \equiv d\zeta/d\xi$ and $\bar\omega_a = {}^1/_2 e_{abc}\bar\tau_\zeta{}^{ib}, \bar\tau_i{}^c$ will be considered functions of the slow variable ξ. The quantity $\Delta(\xi)$ for springs has the meaning of the local length of a coil of the spring.

For brevity later, we assume that explicit mention of the arguments η or ξ in any of the functions means that it satisfies conditions 1) – 3).

Let us present an example. We set $\bar r^i = a^i\zeta, \bar r^i{}_1 = b^i, \bar r_2{}^i = c^i$, where a^i, b^i, c^i are constants of mutually orthogonal unit vectors, and

$$\begin{gathered}\rho^1 = R\cos 2\pi\eta,\ \rho^2 = R\sin 2\pi\eta,\ \rho^3 = 0,\ \eta = \Delta^{-1}\xi \\ \zeta = \varkappa\xi,\ \varkappa = \sin\alpha,\ \alpha \in [-\pi/2,\ \pi/2]\end{gathered} \tag{2.3}$$

Here R is the radius of the spring, the positive quantity Δ is the length of a coil, and α is the pitch of the coil. The quantities R, Δ and α are constants related by the expression $2\pi R = \Delta\cos\alpha$.

The curve Γ defined by (2.1) and (2.3) is a regular spiral line. The ortho reference triad with which the line Γ is provided will be selected to consist of the tangent vector τ_3 and the geometric normal and binormal τ_1, τ_2. Then the components of the orthogonal matrix α_{ab}, representing the projections of the vectors τ_a on the vectors $\bar\tau_b$ are determined by the formulas

$$\begin{aligned}\alpha_{1b} &= (-\cos 2\pi\eta,\ -\sin 2\pi\eta,\ 0)\\ \alpha_{2b} &= (\sin\alpha\sin 2\pi\eta,\ -\sin\alpha\cos 2\pi\eta,\ \cos\alpha)\\ \alpha_{3b} &= (-\cos\alpha\sin 2\pi\eta,\ \cos\alpha\cos 2\pi\eta,\ \sin\alpha)\end{aligned} \tag{2.4}$$

We take the functions I^{ab} as constants. Springs defined in this manner are called regular cylindrical springs, their geometry is given by two parameters, R and α, say. The quantities; ω_a are constant here

$$\omega_1 = 0,\ \omega_2 = 2\pi\Delta^{-1}\cos\alpha,\ \omega_3 = 2\pi\Delta^{-1}\sin\alpha$$

where ω_2 and ω_3 have the respective meanings of the geometric curvature and torsion of the curve Γ, while the function $\bar\omega_a \equiv 0$.

We define the operation of taking the average for any function of the fast variable η by the formula

$$\langle f\rangle = \int_0^1 f(\eta)\,d\eta$$

If $f = f(\eta, \xi)$, then the quantity $\langle f\rangle$ can depend on ξ and satisfy condition 3), i.e., if is a slightly varying function of ξ at distances of the order of Δ.

Without limiting the generality, it can be considered that the functions $\rho^a(\eta, \xi)$ in (2.1) are subject to the constraints

$$\langle \rho^a(\eta, \xi)\rangle = 0. \tag{2.5}$$

If the functions ρ^a do not satisfy (2.5), then its satisfaction could be achieved by making the replacement $\bar r^i \to \bar r^i + \langle\rho^a\rangle\,\bar\tau_a{}^i$.

The relationship $\langle r^i\rangle = \bar r^i$ holds because of (2.5). Therefore, the curve $\bar\Gamma$ yields the middle axis of the spring and it can be interpreted as the axis of an equivalent rod.

The equivalent rod is provided with the ortho reference triad $\bar\tau_a$. If the functions r^i and $\tau_a{}^i$ (and therefore also $\bar r^i$) are known, then the ortho reference triad $\bar\tau_a$ is not reproduced uniquely: the transformation of turn of the vectors $\bar\tau_a$ through a certain angle around the vector $\bar\tau_3$ does not change $r^i, \tau_a{}^i$ and $\bar r^i$

$$\bar\tau_\alpha{}^i \to \bar\tau_\beta{}^i a_\alpha{}^\beta, \bar\tau_3{}^i \to \bar\tau_3{}^i \quad (2.6)$$
$$\alpha_{a\gamma} \to \alpha_{a\delta} a_\gamma{}^\delta, \quad \alpha_{a3} \to \alpha_{a3}$$
$$\rho_\alpha \to \rho_\beta a_\alpha{}^\beta, \quad \rho_3 \to \rho_3$$

Hence, additional conditions uniquely defining the ortho reference triad $\bar\tau_a$ are needed. We take the following constraints as such conditions

$$\rho_2|_{\eta=0} = 0, \quad \rho_1|_{\eta=0} > 0 \quad (2.7)$$

Conditions (2.7) define the orthogonal matrix $a_\beta{}^\alpha(\xi)$ uniquely if these conditions are not satisfied for the ortho reference triad $\bar\tau_a$ taken initially.

The quantities ω_a for springs are calculated from the formulas

$$\omega_a(\eta, \xi) = \Delta^{-1}\theta_a + \varkappa\alpha_{ab}\bar\omega^b + {}^1/_2 e_{abc}\alpha^{bd}_{|\xi}\alpha_d{}^c \quad (2.8)$$
$$\theta_a(\eta, \xi) = {}^1/_2 e_{abc}\alpha^{bd}_{|\eta}\alpha_d{}^c$$

The vertical bar in the subscripts before the η and ξ denote partial differentiation of the functions dependent on the fast and slow variables with respect to η, ξ.

The solution of the problem (1.1)–(1.3) for springs oscillates rapidly. The exact formulation of the problem (1.1)–(1.3) should be replaced by an approximate "average" in which functions of ξ that vary slightly at distances of the order of the coil length Δ appear.

3. The problem of an equivalent rod. It is later shown that for springs the functions $r^i(\xi)$ and $\tau_a{}^i(\xi)$ in the deformed state are representable in the form

$$r^i = \bar r^i(\zeta) + \rho^a(\eta)\bar r_a{}^i(\zeta), \quad \zeta = \zeta(\xi) \quad (3.1)$$
$$\tau_a{}^i = \alpha_a{}^b(\eta)\bar\tau_b{}^i, \quad \eta = \eta(\xi)$$

The functions ρ^a are of the order of Δ (as $\Delta \to 0$). The functions ρ^a and $\alpha_a{}^b$ depend on $\bar r^i$ and $\bar\tau_a{}^i$ and their first derivatives, however, this is not emphasized in the notation.

The functions $\varkappa = d\zeta/d\xi$ and $\bar\omega_a = {}^1/_2 e_{abc}\bar\tau_{,\zeta}{}^{ib}\bar\tau_i{}^c$ vary slightly at a distance of the order of Δ. The functions $\rho^a(\eta)$ and the orthogonal matrix $\alpha_{ab}(\eta)$ satisfy conditions 1)–3) of Sect. 2 and the constraints (2.5) and (2.7).

The functions $\bar r^i$ and $\bar\tau_a{}^i$ determine the state of the equivalent rod in the deformed position and are found from the condition of stationarity of the functional

$$\int_0^l F d\xi \quad (3.2)$$

in a set of functions $\bar r^i$ and $\bar\tau_a{}^i$ satisfying the constraints

$$d\bar r^i/d\xi = \bar\tau_3{}^i, \quad \bar\tau_a{}^i\bar\tau_{ib} = \delta_{ab} \quad (3.3)$$

Here F is a function of the measure of the deformed of the equivalent rod

$$\bar\Omega_a = \varkappa\bar\omega_a - \varkappa_0\bar\omega_{(0)a} \quad \text{and} \quad \gamma = {}^1/_2((\varkappa/\varkappa_0)^2 - 1),$$

which are quantities characterizing the undeformed state $\varkappa_0$, $\bar\omega_{(0)a}$ and the physical characteristics I^{ab} of the rod from which the spring is coiled.

To calculate F the problem must be solved per coil

$$F = \inf\Phi, \quad \Phi = \langle {}^1/_2 I^{ab}\Omega_a\Omega_b\rangle \quad (3.4)$$
$$\Omega_a = \Delta^{-1}(\theta_a - \theta_{(0)a}) + \alpha_{ab}\bar\Omega^b + (\alpha_{ab} - \alpha_{(0)ab})\varkappa_0\bar\omega^b_{(0)}$$

where the lower bound is taken over all periodic functions $\alpha_{ab}(\eta)$ and $\rho_a(\eta)$ satisfying the constraints

$$\alpha_{ac}\alpha_b{}^c = \delta_{ab} \quad (3.5)$$

$$\alpha_{3a} = \varkappa\delta_{3a} + \Delta^{-1}\rho_{a|\eta} + e_{abc}(\bar\Omega^b + \varkappa_0\bar\omega^b_{(0)})\rho^c \quad (3.6)$$

$$\langle\rho^a\rangle = 0, \quad \rho_2|_{\eta=0} = 0, \quad \rho_1|_{\eta=0} > 0 \quad (3.7)$$

The quantities $\bar\Omega_a$, Δ, $\varkappa_0$ and $\bar\omega_{(0)a}$ in the problem (3.4)–(3.7) are constant parameters, while I^{ab}, $\theta_{(0)a}$ and $\alpha_{(0)ab}$ are given periodic functions of η.

The boundary conditions for the functions $\bar{r}^i$ and $\bar{\tau}_a{}^r$ are found from the calculated functions $\rho^a(\eta)$ and $\alpha_{ab}(\eta)$ and the known boundary values of the functions r^i and $\tau_a{}^i$ from (3.1).

If just one end of the spring is rigidly clamped, while a force F_i and moment M_i are given to the other, then the deformed state of the equivalent rod is determined from the variational equation

$$\delta\int_0^l F d\xi - \delta L = 0 \tag{3.8}$$

$$\delta L = (F_i \delta\bar{r}^i + M_i \bar{\tau}_3{}^i \delta\varphi + M_i \bar{\tau}^{i\alpha}\bar{\tau}^{j\beta} e_{\alpha\beta} \delta\bar{r}_{j,\zeta})|_{\xi=l}$$

$$\delta\varphi = {}^1/{}_2 e^{\alpha\beta} \bar{\tau}_\alpha{}^i \delta\bar{\tau}_{i\beta}$$

where $e^{\alpha\beta}$ is the two-dimensional Levi-Civita symbol.

Let us present an expression for F which is obtained as a result of solving the problem (3.4)–(3.7) in the particular case of a regular cylindrical spring coiled from a circular rod (i.e. $I^{\alpha\beta} = I\delta^{\alpha\beta}$, $I^{\alpha 3} = 0$, $I^{33} = J$)

$$2F = I[(2\pi\Delta^{-1} + \bar{\Omega}_3)\sqrt{1-\varkappa^2} - 2\pi\Delta^{-1}\sqrt{1-\varkappa_0{}^2}]^2 + J[2\pi\Delta^{-1} + \bar{\Omega}_3)\varkappa - 2\pi\Delta^{-1}\varkappa_0]^2 + \frac{IJ}{J - {}^1/{}_2(J-I)\sqrt{1-\varkappa_0{}^2}}\bar{\Omega}_\alpha\bar{\Omega}^\alpha \tag{3.9}$$

This expression is derived under the assumption of smallness of $\bar{\Omega}_\alpha$ and cross terms between $\bar{\Omega}_\alpha$ and $\bar{\Omega}_3$, γ are discarded.

Expressions for F under more general assumptions are obtained later.

It turns out that the energy density of F is not a convex function of the strain characteristics $\bar{\Omega}_3$ and $\varkappa$, and the equivalent rod is apparently the simplest example of a model of a continuous medium with a nonconvex energy density. This circle of questions is not examined here.

4. Asymptotic analysis of the variational problem. Let us derive (3.1)–(3.7). We consider the asymptotic in the small parameter Δ/L, the ratio between the length of a coil of the spring and the characteristic scale of the change in a function in the slow argument ξ.

We seek r^i and $\tau_a{}^i$ as functions of the fast and slow variables: $r^i = r^i(\eta, \xi)$, $\tau_a{}^i = \tau_a{}^i(\eta, \xi)$, where $\eta = \eta(\xi)$ is function introduced in giving the undeformed state of the spring. Substituting $r^i(\eta, \xi)$ in the first equation in (1.1) and extracting the principal components in the asymptotic sense, we obtain $r_{|\eta}{}^i = 0$, i.e., in a first approximation, r^i are functions of just the slow variable

$$r^i = \bar{r}^i(\xi) \tag{4.1}$$

As yet the functions $\tau_a{}^i(\eta, \xi)$ are arbitrary. We represent r^i in the form $r^i = \bar{r}^i + r'^i(\eta, \xi)$, where $r'^i(\eta, \xi)$ are asymptotically small additions. Without limiting the generality, it can be considered that $\langle r'^i\rangle = 0$ (if $\langle r'^i\rangle$ are not zero, then the equality $\langle r'^i\rangle = 0$ can be achieved by redefining $\bar{r}^i$ and r'^i: $\bar{r}^i \to \bar{r}^i + \langle r'^i\rangle$, $r'^i \to r'^i - \langle r'^i\rangle$). It is seen from the first equation in (1.1) that r'^i yield a contribution to the constraint if $r'^i \sim \Delta$. Hence, we assume that $r'^i = O(\Delta)$. Let us fix the functions $\bar{r}^i$ and let us seek the functions $r'^i(\eta, \xi)$ and $\tau_a{}^i(\eta, \xi)$. We first make the following substitution.

We construct the functions $\varkappa(\xi)$, $\zeta(\xi)$ and $\bar{\tau}_3{}^i(\xi)$ by means of the fixed $\bar{r}^i(\xi)$ in conformity with the formulas

$$\varkappa = (\bar{r}_{,\xi}{}^i \bar{r}_{i,\xi})^{1/2} = d\zeta/d\xi, \quad \zeta(0) = 0, \quad \bar{r}_{,\xi}{}^i = \varkappa\bar{\tau}_3{}^i \tag{4.2}$$

We construct arbitrary vectors $\bar{\tau}_a$ such that three vectors $\bar{\tau}_a$ would form an orthonormalized reference triad. We introduce the functions $\rho^a(\eta, \xi)$ and the orthogonal matrix $\alpha_{ab}(\eta, \xi)$ by the equalities

$$r'^i = \bar{\tau}_a{}^i \rho^a(\eta, \xi), \quad \tau_a{}^i = \bar{\tau}^i{}_b \alpha_a{}^b(\eta, \xi) \tag{4.3}$$

If the constraint (2.7) is imposed on $\rho^a(\eta, \xi)$, then the vectors $\bar{\tau}_a{}^i$ are determined uniquely by using the transformation (2.6). Therefore, the functions $\bar{\tau}_a{}^i(\xi)$, $\alpha_{ab}(\eta, \xi)$ and $\rho^a(\eta, \xi)$ satisfying the constraints (3.3), (3.5) and (3.7), are defined uniquely by means of given $r'^i(\eta, \xi)$, $\tau_a{}^i(\eta, \xi)$ and $\bar{r}^i(\xi)$. Conversely, the functions r'^i and $\tau_a{}^i$ are reproduced by means of the functions α_{ab} and ρ^a and the formulas (4.3).

Now, let us fix the functions $\bar{r}^i$ and the orthogonal matrix $\bar{\tau}_a{}^i$ that satisfy the relationship (4.2), and we seek the functions α_{ab} and ρ^a that satisfy the constraints (3.5) and (3.7). We substitute (4.3) into the functional (1.3) and the constraint (1.1). We discard the derivatives $\rho_{a|\xi}$, $\alpha_{ab|\xi}$ and $\alpha_{(0)ab|\xi}$ in a first approximation as compared with the quantities $\Delta^{-1}\rho_{a|\eta}$, $\Delta^{-1}\alpha_{ab|\eta}$ and $\Delta^{-1}\alpha_{(0)ab|\eta}$.

Since for any function $f(\eta, \xi)$ the equality

$$\int_0^l f(\eta(\xi), \xi)\, d\xi = \int_0^l \langle f \rangle\, d\xi$$

is valid in the limit $\Delta/L \to 0$, then from the condition of stationarity of the rod energy we obtain the problem for the coil (3.4)– (3.7).

After the problem for the coil has been solved, the functions $\bar{r}^i(\xi)$ and $\bar{\tau}_a{}^i(\xi)$ are determined from the variational problem (3.2), (3.3).

5. Solution of the problem for the coil. We henceforth limit ourselves to the consideration of springs which form a spiral line in the undeformed state (see the example in Sect.2), here

$$\bar{\omega}_{(0)a} = 0 \tag{5.1}$$

i.e., the equivalent rod in the undeformed state is a line and not twisted.

In contrast to the example in Sect.2 in which the vectors τ_α coincided with the geometric normal and binormal, it is later assumed that they can be turned around the vector τ_3 a certain angle φ_0. The corresponding functions $\rho^a_{(0)}$ and $\alpha^{ab}_{(0)}$ have the form

$$\rho^1_{(0)} = R_0 \cos 2\pi\eta, \quad \rho^2_{(0)} = R_0 \sin 2\pi\eta, \quad \rho^3_{(0)} = 0 \tag{5.2}$$

$$\begin{aligned}
&\alpha^{11}_{(0)} = -\cos\varphi_0 \cos 2\pi\eta - \varkappa_0 \sin\varphi_0 \sin 2\pi\eta \\
&\alpha^{12}_{(0)} = -\cos\varphi_0 \sin 2\pi\eta + \varkappa_0 \sin\varphi_0 \cos 2\pi\eta, \quad \alpha^{13}_{(0)} = -k_0 \sin\varphi_0 \\
&\alpha^{21}_{(0)} = \sin\varphi_0 \cos 2\pi\eta + \varkappa_0 \cos\varphi_0 \sin 2\pi\eta \\
&\alpha^{22}_{(0)} = \sin\varphi_0 \sin 2\pi\eta - \varkappa_0 \cos\varphi_0 \cos 2\pi\eta, \quad \alpha^{23}_{(0)} = k_0 \cos\varphi_0 \\
&\alpha^{31}_{(0)} = -k_0 \sin 2\pi\eta, \quad \alpha^{32}_{(0)} = k_0 \cos 2\pi\eta, \quad \alpha^{33}_{(0)} = \varkappa
\end{aligned} \tag{5.3}$$

Here $\varkappa_0$ and k_0 are connected by the relationship $k_0 = \sqrt{1 - \varkappa_0^2}$ while the length of the coil Δ is expressed by the formula $\Delta = 2\pi R_0/k_0$. If we set $\varphi_0 = 0,\ k_0 = \cos\alpha,\ \varkappa_0 = \sin\alpha$ in (5.3), then the functions $\rho^a_{(0)}$ and $\alpha^{ab}_{(0)}$ will go over into the corresponding functions from the example in Sect.2.

The class of springs (5.1)– (5.3) is given by three parameters, for instance the quantities $\varkappa_0$, R_0 and φ_0, which can be slowly varying functions of ξ. This class of springs is basic in engineering applications.

The quantities $\theta^a_{(0)}$ are independent of the fast variable η and are given by the formulas

$$\theta^1_{(0)} = -2\pi k_0 \sin\varphi_0, \quad \theta^2_{(0)} = 2\pi k_0 \cos\varphi_0, \quad \theta^3_{(0)} = 2\pi\varkappa_0 \tag{5.4}$$

The problem for a coil is formulated as a problem to find the minimum of the functional

$$2\Phi = \int_0^1 I^{ab} [\Delta^{-1}(\theta_a - \theta_{(0)a}) + \alpha_{ac}\bar{\Omega}^c][\Delta^{-1}(\theta_b - \theta_{(0)b}) + \alpha_{bc}\bar{\Omega}^c]\, d\eta \tag{5.5}$$

$$\theta_a = {}^1\!/_2 e_{abc} \alpha^{bd}_{|\eta} \alpha_d{}^c, \quad \bar{\Omega}^a = \varkappa \bar{\omega}^a$$

in a set of all periodic functions $\alpha_{ab}(\eta)$ and $\rho^a(\eta)$ satisfying the constraints (3.5)– (3.7), where the constraint (3.6) takes the following form for the class of springs under consideration

$$\alpha_{3a} = \varkappa\delta_{3a} + \Delta^{-1}\rho_{a|\eta} + e_{abc}\bar{\Omega}^b\rho^c \tag{5.6}$$

The solution of the problem for a coil depends on four parameters $\bar{\Omega}_a$ and $\varkappa$ characterizing the strain of the equivalent rod, and three parameters $\varkappa_0$, R_0 and φ_0 (in whose terms $\theta^a_{(0)}$ and Δ are expressed), that carry information about the undeformed state. To obtain and Euler equation of the variational problem (5.5), (3.5), (3.7) and (5.6), the functional (5.5) in which the constraints (3.5), (3.7), (5.6) have been added with appropriate Lagrange multipliers, must be varied. Nonlinear equations with constant coefficients are here obtained. For $\bar{\Omega}_\alpha = 0$ their exact solution can be obtained. In this solution the functions ρ^a and α_{ab} are given by (5.2) and (5.3), in which $\varkappa$ and $R = k(2\pi\Delta^{-1} + \bar{\Omega}_3)^{-1}$ $(k \equiv \sqrt{1-\varkappa^2})$ must be substituted in place of $\varkappa_0$ and R_0, and the root φ of the equation

$$I^{\alpha a}[(2\pi\Delta^{-1} + \bar{\Omega}_3)\, m_a - \Delta^{-1}\theta_{(0)a}]\, m^\beta e_{\alpha\beta} = 0 \tag{5.7}$$

$$m_1 = -k\sin\varphi, \quad m_2 = k\cos\varphi, \quad m_3 = \varkappa$$

in place of φ_0.

Equation (5.7) is a transcendental equation in φ and determines φ as a function of $\varkappa$ and $\overline{\Omega}_3$. In the general case it has two to four roots. The root to which it is possible to arrive by varying $\varkappa$ and $\overline{\Omega}_3$ continuously from the initial value of φ_0 for $\varkappa = \varkappa_0$ and $\overline{\Omega}_3 = 0$, must be selected. An analysis of (5.7) shows that this condition extracts the root in a unique manner. The solution of equation (5.7) depends substantially on the tensor I^{ab} and the value of φ_0. For instance, if the tensor I^{ab} is diagonal and $\varphi_0 = {}^1/_2\pi n$ $(n = 0, 1, 2, \ldots)$, then there is a root $\varphi = \varphi_0$. If meanwhile $I^{\alpha\beta} = I\delta^{\alpha\beta}$, then (5.7) has the root $\varphi = \varphi_0$ for any φ_0. The cases listed are basic for applications.

Let the functions ρ_a and α_{ab} be denoted by $\rho_a{}^*$ and $\alpha_{ab}{}^*$ in the solution obtained. We shall seek additions to the solutions $\rho_a{}^*$ and $\alpha_{ab}{}^*$ that are linear in $\overline{\Omega}_\alpha$. We represent the required quantities ρ_a and α_{ab} in the form

$$\rho_a = \rho_a{}^* + u_a, \quad \alpha_{ab} = \alpha_{db}{}^* (\delta_a{}^d + e_a{}^{dc} \nu_c) \tag{5.8}$$

where $u_a(\eta)$ and $\nu_a(\eta)$ are new required functions.

We substitute (5.8) into the functional (5.5) and the constraints (3.5), (3.7) and (5.6), and we retain the principal terms in $\overline{\Omega}_\alpha$. The principal terms in the functional Φ will be quadratic in $\overline{\Omega}_\alpha$ since the linear terms vanish because of the Euler equations for $\rho_a{}^*$, $\alpha_{ab}{}^*$. It is sufficient to keep only linear terms in the constraints since linear corrections in $\overline{\Omega}_\alpha$ to $\rho_a{}^*$ and $\alpha_{ab}{}^*$ are sought. Then to determine u_a and ν_a we obtain a variational problem to find the minimum of the functional

$$\begin{aligned} 2\Phi_1 = \langle I^{ab} [\Delta^{-1}\nu_{a\,|\,\eta} + (2\pi\Delta^{-1} + \overline{\Omega}_3)\, e_{acd} m^c \nu^d + \overline{\Omega}^\beta \alpha^*_{a\beta}] \times \\ [\Delta^{-1}\nu_{b\,|\,\eta} + (2\pi\Delta^{-1} + \overline{\Omega}_3)\, e_{bcd} m^c \nu^d + \overline{\Omega}^\beta \alpha^*_{b\beta}] + \\ \Delta^{-1} N^a e_{abc} \nu^b_{|\,\eta} \nu^c + H[(\nu_a m^a)^2 - \nu_a \nu^a] + 2N^a \overline{\Omega}^\beta e_a{}^{bc} \alpha^*_{b\beta} \nu_c \rangle \end{aligned} \tag{5.9}$$

$$N^a = I^{ab} [(2\pi\Delta^{-1} + \overline{\Omega}_3)\, m_b - \Delta^{-1}\theta_{(0)\,b}]$$

$$H = (2\pi\Delta^{-1} + \overline{\Omega}_3)\, k^{-2} N^\alpha m_\alpha$$

in the set of all periodic functions $\nu_a(\eta)$ and $u_a(\eta)$ satisfying the constraints

$$\alpha^*_{\beta a} \nu_\gamma e^{\gamma\beta} = \Delta^{-1} u_{a\,|\,\eta} + \overline{\Omega}_3 e_{a3\beta} u^\beta + e_{a\beta}{}^c \overline{\Omega}^\beta \rho_c{}^* \tag{5.10}$$

$$\langle u_a \rangle = 0 \tag{5.11}$$

$$u_2\,|_{\eta=0} = 0 \tag{5.12}$$

Let us note that the functional Φ_1 is invariant relative to the transformation

$$\nu_a \to \nu_a + t m_a \tag{5.13}$$

where t is an arbitrary constant. For given ν_γ satisfying the constraints

$$\langle \alpha^*_{\beta a} \nu_\gamma e^{\gamma\beta} \rangle = 0 \tag{5.14}$$

periodic functions $u_a(\eta)$ satisfying the constraints (5.11) can always be found from (5.10). Then by using the transformation (5.13) (the constraint (5.14) is also invariant relative to this transformation), compliance with the equality (5.12) can be achieved without changing the value of the functional. Hence, the problem for ν_a and u_a can be solved as follows: first the minimum of Φ_1 is sought in all the periodic $\nu_a(\eta)$ (here the minimizing element is determined to the accuracy of the transformation (5.13)), then $u_a(\eta)$ are found from (5.10) and (5.11), after which the constant t is selected from condition (5.12).

The Euler equations for the variational problem (5.9), (5.14) will be linear inhomogeneous differential equations with constant coefficients, where the inhomogeneous terms are a linear combination of sines, cosines, and constants. Hence, ν_a have the form

$$\nu_a = q_a + g_a e^{2\pi i\eta} + \bar{g}_a e^{-2\pi i\eta} \tag{5.15}$$

Here q_a are real, while g_a are complex constants, i is the imaginary unit, and the bar above the complex quantities denotes the complex conjugate. Possible resonance terms are not in formula (5.15) since they do not satisfy the periodicity condition.

We also represent the functions $\alpha_{ab}{}^*$ in an analogous form

$$\alpha^*_{ab} = Q_{ab} + G_{ab} e^{2\pi i\eta} + \overline{G}_{ab} e^{-2\pi i\eta} \tag{5.16}$$

$$Q_{a3} = \psi_{a3}, \quad Q_{\alpha\alpha} = 0, \quad G_{a3} = 0, \quad G_{a\alpha} = {}^1/_2 \psi_a{}^\beta B_{\alpha\beta}$$

$$\psi_{ab} = \alpha^*_{ab}\,|_{\eta=0}, \quad B_{ab} = \delta_{ab} + i e_{ab3}$$

Let us note that the relationship $\psi_{a3} = m_a$ holds.
The second order matrix $B_{\alpha\beta}$ possesses the following properties

$$\bar{B}_{\alpha\beta} = B_{\beta\alpha}, \quad B^{\alpha\beta}B_{\gamma\beta} = 0, \quad B_{\alpha\beta}B^{\beta}{}_{\gamma} = 2B_{\alpha\gamma} \qquad (5.17)$$
$$B_{\alpha\beta}B_{\gamma\delta} = B_{\alpha\delta}B_{\gamma\beta}, \quad B_{\alpha\beta}e^{\alpha}{}_{\gamma} = iB_{\gamma\beta}$$

After substitution of (5.15) and (5.16), the constraints (5.14) result in the relationships

$$e^{\alpha\beta}G_{\alpha\gamma}g_{\beta} + e^{\alpha\beta}\bar{G}_{\alpha\gamma}\bar{g}_{\beta} = 0, \quad e^{\alpha\beta}m_{\alpha}q_{\beta} = 0$$

It can be confirmed that the general solution of these equations is given by the formulas

$$g^{a} = x\delta^{a3} + y\psi^{a\beta}B_{\beta\alpha}\psi^{3\alpha}, \quad q^{a} = tm^{a} + z\delta^{a3} \qquad (5.18)$$

where x and y are complex while z and t are real arbitrary constants.

We substitute (5.18) into (5.9). We obtain a function of x, y, z, t which must be minimized in these variables

$$2\Phi_2 = S_0 + S_1x\bar{x} + S_2y\bar{y} + (Kx\bar{y} + \bar{K}\bar{x}y) + (\bar{X}x + X\bar{x}) + (Yy + \bar{Y}\bar{y}) + S_3z^2 \qquad (5.19)$$

Here

$$S_0 = S\bar{\Omega}_{\alpha}\bar{\Omega}^{\alpha}, \quad S_1 = (2\pi/\Delta)^2 J_{ab}\psi_{3c}\psi_{3d}b^{ac}b^{db} - Hk^2$$
$$S_2 = (2\pi k/\Delta)^2 J_{\alpha\beta}b^{\alpha\gamma}b^{\delta\beta}B_{\gamma\delta} - 4\pi\Delta^{-1}k^2N^{a}m_a - Hk^2$$
$$K = (2\pi/\Delta)^2 J_{a\alpha}\psi_{3c}b^{ac}b^{\beta\alpha}B_{\gamma\beta}\psi^{3\gamma} + 2\pi\Delta^{-1}N^{\alpha}\psi_{3\alpha} - Hk^2$$
$$S_3 = {}^1/_2(2\pi\Delta^{-1} + \bar{\Omega}_3)J_{\alpha\beta}e^{\alpha\gamma}e^{\beta\delta}\psi_{3\gamma}\psi_{3\delta} - Hk^2$$
$$X = X_{\alpha}B^{\beta\alpha}\bar{\Omega}_{\beta}, \quad Y = Y_{\alpha}B^{\alpha\beta}\bar{\Omega}_{\beta}$$
$$S = {}^1/_4 J_{\alpha}{}^{\alpha}, \quad J_{ab} = I^{cd}\psi_{ca}\psi_{db}$$
$$b_{ab} = \delta_{ab} + i(1 + \bar{\Omega}_3\Delta(2\pi)^{-1})e_{ab3}$$
$$X_{\alpha} = -i\pi\Delta^{-1}J_{a\alpha}\psi_{3c}b^{ca} - i{}^1/_2k^{-1}N^{\beta}m_{\beta}\psi_{3\alpha}$$
$$Y_{\alpha} = i\pi\Delta^{-1}J_{\alpha\beta}b^{\beta\gamma}B_{\gamma\delta}\psi^{3\delta}$$

The t does not enter (5.19), as should be because of the invariance of Φ_1 relative to the transformation (5.13). The constant t is evaluated after the determination of u_a as has been described above.

We find x, y, z from the stationarity conditions for the function Φ_2

$$x = a^{-1}(\bar{Y}\bar{K} - S_2X), \quad y = a^{-1}(XK - S_1Y)$$
$$z = 0, \quad a = S_1S_2 - K\bar{K}$$

The stationary value of the function Φ_2 has the form ${}^1/_2E\bar{\Omega}_{\alpha}\bar{\Omega}^{\alpha}$ where

$$E = S - a^{-1}[S_1Y_{\alpha}\bar{Y}^{\alpha} + S_2X_{\alpha}\bar{X}^{\alpha} - (KX_{\alpha}Y^{\alpha} + \bar{K}\bar{X}_{\alpha}\bar{Y}^{\alpha})] \qquad (5.20)$$

The coefficient E depends on $\bar{\Omega}_3$, $\varkappa$ and characteristics of spring. This dependence is complex in the general case. The expression for E simplifies substantially if cross effects between the bending and torsion-tension of the equivalent rod are neglected, i.e., $\bar{\Omega}_3 = 0$ and $\varkappa = \varkappa_0$ must be set in the expression for E. The simplification is related to the fact that H and N_a vanish in this case while $b^{ab} = B^{ab}$, and therefore, (5.17) can be applied to $b^{\alpha\beta}$. We consequently obtain for E (I_{ab}^{-1} is the inverse matrix to I^{ab})

$$E|_{\bar{\Omega}_3=0,\, \varkappa=\varkappa_0} = E_0 = 2(I_a^{-1a} - I_{ab}^{-1}m_{(0)}^{a}m_{(0)}^{b})^{-1} \qquad (5.21)$$

6. Equations of the equivalent rod.

Thus, the expression

$$2F = I^{ab}[(2\pi\Delta^{-1} + \bar{\Omega}_3)m_a - \Delta^{-1}\theta_{(0)a}] \times \qquad (6.1)$$
$$[(2\pi\Delta^{-1} + \bar{\Omega}_3)m_b - \Delta^{-1}\theta_{(0)b}] + E\bar{\Omega}_{\alpha}\bar{\Omega}^{\alpha}$$
$$m_1 = -\sqrt{1-\varkappa^2}\sin\varphi, \quad m_2 = \sqrt{1-\varkappa^2}\cos\varphi, \quad m_3 = \varkappa$$

is obtained for F.

Here φ is a function of $\bar{\Omega}_3$ and $\varkappa$ given implicitly by (5.7), and E is given by (5.20).

Let us note that (5.7) can be obtained by equating the partial derivative of F with respect to φ to zero for $\bar{\Omega}_{\alpha} = 0$.

The Euler equations following from the problem (3.2), (3.3), (3.8) have the form

$$\overline{M}^a_{,\zeta} = -e^{abc}\overline{\omega}_b\overline{M}_c + e^{ab3}\overline{\tau}_{ib}F^i, \ \overline{M}^a\overline{\tau}_a{}^i|_{\xi=l} = M^i \tag{6.2}$$
$$T = \overline{\tau}^{i3}F_i, \quad \overline{r}^i_{,\zeta} = \overline{r}^{i3}$$
$$\overline{\omega}_a = {}^1/_2 e_{abc}\overline{\tau}^{ib}_{,\zeta}\overline{\tau}_i{}^c, \quad \overline{\tau}^i_a\overline{\tau}_{ib} = \delta_{ab}$$

The moment $\overline{M}_a$ and the tensile force T are determined by the equations of state

$$\overline{M}_a = \partial F/\partial\overline{\Omega}^a, \quad T = \partial F/\partial\varkappa \tag{6.3}$$

If $E = E_0$, then the equations of state take the form

$$\overline{M}^\alpha = E_0\overline{\Omega}^\alpha, \quad \overline{M}^3 = I^{ab}m_a[(2\pi\Delta^{-1} + \overline{\Omega}_3)m_b - \Delta^{-1}\theta_{(0)b}] \tag{6.4}$$
$$T = (2\pi\Delta^{-1} + \overline{\Omega}_3)[I^{\alpha\beta}m_\beta\varkappa(1-\varkappa^2)^{-1} + I^{a3}] \times$$
$$[(2\pi\Delta^{-1} + \overline{\Omega}_3)m_a - \Delta^{-1}\theta_{(0)a}]$$

7. Examples. 1°. We examine the problem of tension-torsion of a spring ($F^i = M^i = 0$ for $i = 1,2$). We represent the required functions $\overline{\tau}^{ia}$ and $\overline{r}^i$ in the form

$$\overline{\tau}^{i1} = (\cos\theta(\xi), \ -\sin\theta(\xi), \ 0)$$
$$\overline{\tau}^{i2} = (\sin\theta(\xi), \ \cos\theta(\xi), \ 0)$$
$$\overline{\tau}^{i3} = \delta^{i3}, \ \overline{r}^i = \zeta(\xi)\,\delta^{i3}$$

We consider $\theta(\xi) = 0$ and $\zeta(\xi) = \zeta_0(\xi)$ in the undeformed state.

Then the equations for the equivalent rod go over into the following two relationships for the functions $\theta(\xi)$ and $\zeta(\xi)$:

$$\overline{M}^3 = M^3, \quad T = F^3 \tag{7.1}$$

where $\overline{M}^3$ and T are determined by (6.4), where $\overline{\Omega}_3 = -\theta_{,\xi}$, $\varkappa = \zeta_{,\xi}$.

Equations (7.1) determine the functions $\theta(\xi)$ and $\zeta(\xi)$ to the accuracy of the solid motion of the equivalent rod, which is fixed, say, by giving the location of one of the ends of the spring. Knowing ρ_a and α_{ab} the shape of the spring can be determined from (3.1).

The solution of (7.1) reduces to solving a system of three finite equations in $\theta_{,\xi}$, $\zeta_{,\xi}$, φ and a simple quadrature.

If the spring is a regular cylinder, i.e., the quantities $\theta_{(0)a}$, I^{ab} and Δ are constants, then it can be verified that (7.1) are equivalent to the exact equations of the initial problem (1.1) - (1.5). This is natural since in this case the quantity Δ/L used as a small parameter in the variational-asymptotic method, is zero.

2°. Let us consider the problem of pure bending of a spring by forces applied to its ends, which produce moments equal in absolute value to M. In this case it can be considered that $F^i = 0$, $M^2 = M$, $M^1 = M^3 = 0$. We represent the required functions $\overline{\tau}^{ia}(\xi)$ and $\overline{r}^i(\xi)$ in the form

$$\overline{\tau}^{i1} = (\sin\theta(\xi), \ \ 0, -\cos\theta(\xi)), \ \overline{r}^1 = \int_0^\xi \varkappa_0(s)\cos\theta(s)\,ds \tag{7.2}$$
$$\overline{\tau}^{i2} = (0, 1, 0), \quad \overline{r}^2 = 0$$
$$\overline{\tau}^{i3} = (\cos\theta(\xi), 0, \sin\theta(\xi)), \quad \overline{r}^3 = \int_0^\xi \varkappa_0(s)\sin\theta(s)\,ds$$

where $\theta(\xi)$ is the new required function that equals zero in the undeformed state.

Then (6.2) and (6.4) reduce to the relationship $\theta_{,\xi} = M/E_0$ from which

$$\theta(\xi) = M\int_0^\xi \frac{ds}{E_0(s)} + \theta(0) \tag{7.3}$$

If $E_0 = \mathrm{const}$ (for instance, for a regular cylindrical spring), then the state of strain of the equivalent rod is mapped by the arc of a circle of radius

$$\overline{R} = \varkappa_0 E_0/M \tag{7.4}$$

When the tensor I^{ab} is of diagonal form, where $I^{\alpha\beta} = I\delta^{\alpha\beta}$, $I^{33} = J$, then (7.4) goes over into the following expression

$$\overline{R} = \sin\alpha_0\,\{[1/I - {}^1/_2\,(1/J - 1/I)\cos\alpha_0]\ M\}^{-1}$$

where α_0 is the pitch of the coil of the spring in the undeformed state. This formula is in agreement with the expression obtained by S.P. Timoshenko /2/.

If $I^{ab} = I\delta^a$, then (7.2)–(7.4), (3.1) yield the exact solution of the initial problem (1.1)–(1.5) obtained in /10/. This fact has the same explanation as the analogous result in the preceding example.

8. Physically linear theory of an equivalent rod. Let us expand the energy of the equivalent rod (6.1) into a series in the measures of the strain $\bar{\Omega}_a$ and $\gamma = {}^1/_2\,((\varkappa/\varkappa_0)^2 - 1)$ and let us discard all terms of order higher than the second, We obtain

$$\begin{aligned}
&2F = A\gamma^2 + B\,(\bar{\Omega}_3)^2 + 2C\bar{\Omega}_3\gamma + E_0\bar{\Omega}_\alpha\bar{\Omega}^\alpha \\
&A = (2\pi\Delta^{-1})^2[-(G\varkappa_0 - G_3)\,D^{-1} + I_{33} + \varkappa_0{}^2 I - \\
&\quad 2\varkappa_0 I_3]\,\varkappa_0{}^2\,(1 - \varkappa_0{}^2)^{-2} \\
&B = I - G^2D^{-1}, \quad C = (2\pi\Delta^{-1})[(G\varkappa_0 - G_3)\,GD^{-1} + I_3 - \\
&\quad \varkappa_0 I]\,\varkappa_0\,(1 - \varkappa_0{}^2)^{-1} \\
&G = I_{a\alpha}m_{(0)}{}^a e_\beta{}^\alpha m_{(0)}{}^\beta, \quad G_3 = I_{3\alpha}e_\beta{}^\alpha m_{(0)}{}^\beta \\
&D = I_{\alpha\beta}e_\gamma{}^\alpha e^\beta m_{(0)}{}^\gamma m_{(0)}{}^\delta, \quad I = I_{ab}m_{(0)}{}^a m_{(0)}{}^b, \quad I_3 = I_{3a}m_{(0)}{}^a
\end{aligned} \tag{8.1}$$

which is the energy of a physically linear theory of an equivalent rod.

If the tensor I_{ab} is diagonal and the diagonal elements equal the quantities I_1, I_2 and J, then the coefficients A, B, C and E_0 in the energy (8.1) have the form

$$\begin{aligned}
&A = (2\pi\Delta^{-1})^2\,\varkappa_0{}^2\,[J - \bar{I}\varkappa_0{}^2\,(1 - \varkappa_0{}^2)^{-1}] \\
&B = J\varkappa_0{}^2 + \bar{I}\,(1 - \varkappa_0{}^2), \quad C = 2\pi\Delta^{-1}\varkappa^2\,(J - \bar{I}) \\
&E_0 = 2\,[(1/I_1 + 1/I_2) + (1/J - 1/\bar{I})(1 - \varkappa_0{}^2)]^{-1} \\
&\bar{I} = I_1I_2/(I_1\cos^2\varphi_0 + I_2\sin^2\varphi_0)
\end{aligned}$$

In the geometrically linear theory, it is necessary to take

$$\bar{\Omega}_\alpha = -e_{\alpha\beta}\frac{d^2v^\beta}{d\zeta^2}, \quad \bar{\Omega}_3 = \frac{d\Theta}{d\zeta}, \quad \gamma = \frac{dv^3}{d\zeta}$$

for $\bar{\Omega}_a$ and γ, where v^i is the displacement of the equivalent rod, and Θ is its torsion. We arrive here at the linear theory of springs constructed in /6/.

REFERENCES

1. TIMOSHENKO S.P. and LESSELS D.S., Applied Theory of Elasticity. GOSTEKHIZDAT, Leningrad, 1931.
2. TIMOSHENKO S.P., Strength of Materials, Vol.2, NAUKA, Moscow, 1965.
3. PONOMAREV S.D., BIDERMAN V.L., LIKHAREV K.K., MAKUSHIN V.M., MALININ N.N. and FEODOS'EV V.I., Strength Analyses in Machine Construction. Vol.1, Mashgiz, Moscow, 1956.
4. KHVINGIIA M.V., Spring Vibrations. MASHINOSTROENIE, Moscow, 1969.
5. ANDREEVA L.E., Elastic Elements of Instruments. Mashgiz, Moscow, 1962.
6. SUTYRIN V.G., On the equations of the theory of springs, Dokl. Akad. Nauk SSSR, Vol.254, No.1, 1980.
7. KIRCHHOFF G.R., Mechanics. Lectures on Mathematical Physics. Izd. Akad. Nauk SSSR, 1962.
8. THOMSON W. and TAIT P., Treatise on Natural Philosophy, Vol.1, Cambridge Univ. Press, 1890..
9. CHERNYSHEV N.A., Nonlinear theory of elastic deformations of cylindrical coil springs. In: Strength Analyses, Vol.3, Mashgiz, Moscow, 1958.
10. COSTELLO G.A., Large deflection of a helical spring due to bending, J. Eng. Mech., Div. Proce. Amer. Soc. Civ. Eng., Vol.103, No.3, 1977.
11. BERDICHEVSKII V.L., Spatial averaging of periodic structures, Dokl. Akad. Nauk SSSR, Vol. 222, No.3, 1975.
12. BERDICHEVSKII V.L., Variational-asymptotic method of constructing a theory of shells, PMM, Vol.43, No.4, 1979.
13. BERDICHEVSKII V.L., On equations of the theory of anisotropic inhomogeneous rods. Dokl. Akad. Nauk SSSR, Vol.228, No.3, 1976.
14. BERDICHEVSKII V.L., On the energy of an elastic rod, PMM, Vol.45, No.4, 1981.

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